

# **A Quantum Structure Description of the Liar Paradox**

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In this article we propose an approach that models the truth behavior of cognitive entities (i.e., sets of connected propositions) by taking into account in a very explicit way the possible influence of the cognitive person (the one that interacts with the considered cognitive entity). Hereby we specifically apply the mathematical formalism of quantum mechanics because of the fact that this formalism allows the description of real contextual influences, i.e., the influence of the measuring apparatus on the physical entity. We concentrated on the typical situation of the liar paradox and have shown that (1) the true–false state of this liar paradox can be represented by a quantum vector of the nonproduct type in a finite-dimensional complex Hilbert space and the different cognitive interactions by the actions of the corresponding quantum projections, (2) the typical oscillations between false and true—the paradox—is now quantum dynamically described by a Schrödinger equation. We analyze possible philosophical implications of this result.

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## **1. INTRODUCTION**

The liar paradox is the oldest semantic paradox in the literature. In its simplest form it can be traced back to Eubulides—a pupil of Euclid—and to the Cretan Epimenides. Since the Greeks different alternative forms of the liar have emerged, including variations of the one-sentence paradox (the simplest form of the liar) and paradoxes of two or more sentences. The two-sentence paradox is known as the postcard paradox of Jourdain, and goes back to Buridan in 1300. On one side of a postcard we read, ‘the sentence on the other side of this card is true,’ and on the other side we read, ‘the sentence on the other side of this card is false.’

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In this paper we will not work with the original forms of the paradox, but in a version in which we use an index or sentence pointer followed by the sentence at which this index points:

*Single Liar*

- (1) Sentence (1) is false

*Double Liar*

- (1) Sentence (2) is false  
(2) Sentence (1) is true

## 2. APPLYING THE QUANTUM MECHANICAL FORMALISM

The theories of chaos and complexity have shown that similar patterns of behavior can be found in very different layers of reality. The success of these theories demonstrates that interesting conclusions about the nature of reality can be inferred from the encountered structural similarities of dynamical behavior in different regions of reality. Chaos and complexity theories are, however, deterministic theories that do not take into account the fundamental contextuality that is introduced by the influence of the act of observation on the observed. Most of the regions of reality are highly contextual (e.g., the social layer, the cognitive layer, the pre-material quantum layer), unlike the material layer of reality where contextuality is minimal. In this sense it is strange that no attempts have been undertaken to find similarities using contextual theories, such as quantum mechanics, in the different regions of reality. The study in this paper should be classified as such an attempt, and is part of one of the projects in our center focusing on the layered structure of reality (Clea Research Project, 1997–; Aerts, 1994; Aerts, 1998a)

We justify the use of the mathematical formalism of quantum mechanics to model context-dependent entities because a similar approach has already been developed for the situation of an opinion poll within the social layer of reality (Aerts, 1998b; Aerts and Aerts, 1994, 1997; Aerts *et al.*, 1999). In such an opinion poll specific questions are put forward that introduce a real influence of the interviewer on the interviewee, such that the situation is contextual. It is shown explicitly in Aerts and Aerts (1994, 1997) that the probability model that results in this situation is of a quantum mechanical nature.

By means of a model we will present the liar—one sentence—or the double liar—a group of sentences—as one entity that we consider to ‘exist’ within the cognitive layer of reality. The existence is being expressed by the

possibility of influencing other cognitive entities, and by the different states that it can be in. Indeed it has been shown that the concept of entity can be introduced rigorously and founded on the previously mentioned properties. In this way we justify the present use (Aerts, 1992).

### 3. MEASURING COGNITIVE ENTITIES: MODELING TRUTH BEHAVIOR

In this section we will explore the context dependence of cognitive entities like the liar paradox. We introduce the explicit dependence of the truth and falsehood of a sentence on the cognitive interaction with the cognitive person. Reading a sentence or in other words ‘making a sentence true or false,’ will be modeled as ‘performing a measurement’ on the sentence within the cognitive layer of reality. This means that in our description a sentence within the cognitive layer of reality is ‘in general’ neither true nor false. The ‘state true’ and the ‘state false’ of the sentence are ‘eigenstates’ of the measurement. During the act of measurement the state of the sentence changes in such a way that it is true or it is false. This general ‘neither true nor false state’ will be called a superposition state in analogy with the quantum mechanical concept. We shall see that it is effectively a superposition state in the mathematical sense after we have introduced the complex Hilbert space description.

We proceed operationally as follows. Before the cognitive measurement (this means before we start to interact with the sentence, read it, and make a hypothesis about its truth or falsehood) the sentence is considered to be neither true nor false and hence in a superposition state. If we want to start to analyze the cognitive inferences entailed, we make one of the two possible hypotheses that it is true or that it is false. The making of one of these two hypotheses—this is part of the act of measurement—changes the state of the sentence to one of the two eigenstates—true or false. As a consequence of the act of measurement the sentence becomes true or false (is in the state true or false) within the cognitive entity of which the sentence is part. This change influences the state of this complete cognitive entity. We will see that if we apply this approach to the double liar, the change of state puts into work a dynamic process that we can describe by a Schrödinger equation. We have to consider three situations:

$$\begin{array}{l}
 \text{A} \left\{ \begin{array}{l} (1) \text{ Sentence (2) is false} \\ (2) \text{ Sentence (1) is true} \end{array} \right. \\
 \\
 \text{B} \left\{ \begin{array}{l} (1) \text{ Sentence (2) is true} \\ (2) \text{ Sentence (1) is true} \end{array} \right.
 \end{array}$$

$$C \begin{cases} (1) & \text{Sentence (2) is false} \\ (2) & \text{Sentence (1) is false} \end{cases}$$

#### 4. THE DOUBLE LIAR: A FULL QUANTUM DESCRIPTION

The resemblance of the truth values of single sentences and the twofold eigenvalues of a spin-1/2 state is used to construct a dynamical representation; the measurement evolution as well as a continuous time evolution are included.

We recall some elementary properties of a spin state. Elementary particles like the electron have a property referred to as an intrinsic angular momentum or spin. The spin of a particle is quantized: upon measurement the particle only exposes a finite number of distinct spin values. For the spin-1/2 particle, the number of spin states is two, commonly referred to as the ‘up’ and ‘down’ states. This two-valuedness can adequately describe the truth function of a liar-type cognitive entity. Such a sentence supposedly is either true or false. The quantum mechanical description, on the other hand, allows a superposition of the ‘true’ and ‘false’ states. This corresponds to our view of allowing cognitive entities before measurement—i.e., reading and hypothesizing—to reside in a nondeterminate state of truth or falsehood. In quantum mechanics such a state  $\Psi$  is described by a compounded superposition of the two states:

$$\Psi = c_{\text{true}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_{\text{false}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The operation of finding whether such a cognitive entity is true or false is done by applying respectively the true-projector  $P_{\text{true}}$  or false-projector  $P_{\text{false}}$ :

$$P_{\text{true}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{\text{false}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

In practice in the context of the cognitive entity, this corresponds to the assignment of either truth or falsehood to a sentence after its reading. In quantum mechanics, the true-measurement on the superposed state  $\Psi$  results in the true state,

$$P_{\text{true}} \Psi = c_{\text{true}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

while the square modulus of the corresponding compounding factor  $c_{\text{true}}$  gives the statistical probability of finding the entity in the true-state. An unequivocal result is therefore not obtained when the superposition does not leave out

one of the states completely, i.e., either  $c_{\text{true}}$  or  $c_{\text{false}}$  is zero. Only in those instances do we attribute to a sentence its truth or falsehood.

The coupled sentences of the two-sentence liar paradox (C), for instance, are precisely described by the so-called ‘singlet state.’ This global state combines, using the tensor product  $\otimes$ , states of sentence 1 with states of sentence 2:

$$\frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

The appropriate true-projectors for sentence one and two are now

$$P_{1,\text{true}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1_2, \quad P_{2,\text{true}} = 1_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The false-projectors are obtained by switching the diagonal elements 1 and 0 on the diagonal of the matrix.

In the same manner the coupled sentences of the liar paradox (B) can be constructed:

$$\frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Our final aim is to describe the real double liar paradox (A) quantum mechanically and even more to show how the true–false cycle originates from the Schrödinger time evolution of the appropriate initial state. The description of this system necessitates the coupled Hilbert space  $C^4 \otimes C^4$ , a larger space than for the previous systems. In this case the truth and falsehood values from measurement and semantic origin must be discerned; the dimension for each sentence therefore must be 4.

The initial unmeasured state, i.e.,  $\Psi_0$  of the real double liar paradox is

$$\frac{1}{2} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Each next term in this sum is actually the consecutive state which is reached in the course of time when the paradox is read through. This can be easily verified by applying the appropriate truth-operators:

$$P_{1,\text{true}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1_2, \quad P_{2,\text{true}} = 1_1 \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The projectors for the false-states are constructed by placing the 1 on the final diagonal place.

The explicit construction of the unitary evolution operator is accomplished through an intermediary equivalent representation in  $C^{16}$ . The complex space  $C^4 \otimes C^4$  is isomorphic to  $C^{16}$ . In this aim the basis of the  $C^{16}$  is constructed as ( $i$  and  $j$  from 1 to 4):

$$e_i \otimes e_j = e_{\kappa(i,j)} \quad \text{and} \quad \kappa(i,j) = 4(i-1) + j$$

In  $C^{16}$  the unmeasured state  $\Psi_0$  is then given by

$$\Psi_0 = \frac{1}{2} \{e_{10} + e_8 + e_{13} + e_3\}$$

The 4 by 4 submatrix  $U_D$  of the discrete unitary evolution operator, which describes the time evolution at instants of time when a sentence has changed truth value, is

$$U_D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In order to obtain a description at every instant of time, a procedure of diagonalization on the submatrix  $U_D$  was performed, i.e.,  $U_D|_{\text{diag}}$ . From the Schrödinger evolution and Stone's Theorem we obtain

$$H_{\text{sub}}|_{\text{diag}} = i \ln U_D|_{\text{diag}}$$

Now inverting the procedure of diagonalization, we obtain the infinitesimal generator of the time evolution—the submatrix Hamiltonian;

$$H_{\text{sub}} = \begin{pmatrix} -1/2 & -1/2 & (1-i)/2 & (1+i)/2 \\ -1/2 & -1/2 & (1+i)/2 & (1-i)/2 \\ (1+i)/2 & (1-i)/2 & 1/2 & 1/2 \\ (1-i)/2 & (1+i)/2 & 1/2 & 1/2 \end{pmatrix}$$

The submatrix of the evolution operator  $U(t)$ , valid at all times, is then given by the expression

$$U_{\text{sub}}(t) = e^{-iH_{\text{sub}}t}$$

The time evolution operator  $U_{\text{sub}}(t)$  in the 4 by 4 subspace of  $C^{16}$  becomes (modulo a numerical factor  $\frac{1}{4}$  for all elements)

$$\begin{pmatrix} 1 + e^{-it} + e^{it} + e^{2it} & 1 - e^{-it} - e^{it} + e^{2it} \\ 1 - e^{-it} - e^{it} + e^{2it} & 1 + e^{-it} + e^{it} + e^{2it} \\ 1 + ie^{-it} - ie^{it} - e^{2it} & 1 - ie^{-it} + ie^{it} - e^{2it} \\ 1 - ie^{-it} + ie^{it} - e^{2it} & 1 + ie^{-it} - ie^{it} - e^{2it} \\ 1 - ie^{-it} + ie^{it} - e^{2it} & 1 + ie^{-it} - ie^{it} - e^{2it} \\ 1 + ie^{-it} - ie^{it} - e^{2it} & 1 - ie^{-it} + ie^{it} - e^{2it} \\ 1 + e^{-it} + e^{it} + e^{2it} & 1 - e^{-it} - e^{it} + e^{2it} \\ 1 - e^{-it} - e^{it} + e^{2it} & 1 + e^{-it} + e^{it} + e^{2it} \end{pmatrix}$$

The Hamiltonian  $H$  as well as the time-evolution operator  $U(t)$  in  $C^4 \otimes C^4$  is immediately obtained by inverting the basis transformation function  $\kappa$ :

$$H = \sum_{\kappa, \lambda=1}^{16} H_{\text{sub}\kappa(i,j)\lambda(u,v)} O_{iu} \otimes O_{jv}$$

and

$$U(t) = \sum_{\kappa, \lambda=1}^{16} U_{\text{sub}\kappa(i,j)\lambda(u,v)}(t) O_{iu} \otimes O_{jv}$$

with

$$O_{iu} \otimes O_{jv} = \{e_i \cdot e_u^t\} \otimes \{e_j \cdot e_v^t\}$$

For example, the term  $\kappa = 3, \lambda = 10$  of the time evolution operator  $U(t)$  is

$$\frac{1}{4} (1 - ie^{-it} + ie^{-it} - ie^{2it}) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Starting from the initial state  $\Psi_0$  the constructed dynamical evolution leaves the system unchanged;  $\Psi_0$  is a time-invariant state:

$$\Psi_0(t) = \Psi_0$$

As soon as a measurement for truth or falsehood on either of the sentences is made, the dynamical evolution sets off in a cyclical mode, attributing alternatively truth and falsehood to the consecutively read sentences.

The quantum formalism therefore seems an appropriate tool to describe the liar paradox. Could the formalism be applied to more intricate cognitive

entities? Given the procedure we applied—an adaptation of the formalism of two interacting spin-3/2 particles—it is possible to extend the liar paradox to more complex variants of multiple sentences referring to one another in a truth confirming or denying manner. The minimal dimension to represent quantum mechanically such a paradoxical set of  $n$  sentences will not be less than  $2^n$ . The exact dimension of the appropriate Hilbert space depends on the specific  $n$ -sentence liar paradox described.

## 5. CONCLUSION

We analyzed how cognitive entities behave by using the formalism of quantum mechanics where the influence of the cognitive observer on the cognitive entity can be taken into account. In the same way as we described the double liar we can also represent the  $n$ -dimensional liar. The vector in the Hilbert space that we used to represent the state of the double liar is an eigenvector of the Hamiltonian of the system. This shows that we can consider the double liar as a cognitive entity without being measured on as an invariant of the time evolution. Once a measurement—a cognitive act—on one of the subelements is performed, the whole cognitive entity changes into a state that is no longer an eigenstate of the Hamiltonian. After this measurement this state will start to change dynamically in the typical way of the liar paradox, sentences becoming true and false, and staying constantly coupled. This behavior is exactly described by the Schrödinger equation that we have derived. In this way we have given a description of the internal dynamics within self-referring cognitive entities such as the liar paradox. Our aim is to develop this approach further and to analyze how we can describe other examples of cognitive entities. We also want to analyze in further research how this result can be interpreted within a general scheme that connects different layers of reality structurally. Some profound philosophical questions, still very speculative at this stage of our research, but certainly stimulating, can be put forward: e.g., Can we learn something about the nature and origin of dynamical change by considering this example of the liar paradox? Could the cognitive layer be considered being in a very early structuring stage, such that we trace down very primitive dynamical and contextual processes that could throw some light on primitive dynamical and contextual processes encountered in the prematerial layer (e.g., spin processes)? Apart from these speculative but stimulating philosophical questions, we also would like to investigate further how our quantum mechanical model for the cognitive layer of reality could be an inspiration for the development of a general interactive logic that can take into account more subtle dynamical and contextual influences than just those of the cognitive person on the truth behavior of the cognitive entities.



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